

ON THE RELATIVE COHEN-MACAULAY MODULES

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ABSTRACT. Let R be a commutative Noetherian local ring and let \mathfrak{a} be a proper ideal of R . A non-zero finitely generated R -module M is called relative Cohen-Macaulay with respect to \mathfrak{a} if there is precisely one non vanishing local cohomology modules $H_{\mathfrak{a}}^i(M)$ of M . In this paper, as a main result, it is shown that if M is a Gorenstein R -module, then $H_{\mathfrak{a}}^i(M) = 0$ for all $i \neq c$ where $c = \text{ht}_M \mathfrak{a}$ is completely encoded in homological properties of $H_{\mathfrak{a}}^c(M)$, in particular in its Bass numbers. Notice that, this result provides a generalization of a result of M. Hellus and P. Schenzel which has been proved before, as a main result, in the case where $M = R$.

1. INTRODUCTION

Throughout this paper, R is a commutative Noetherian ring, \mathfrak{a} is a proper ideal of R and M is an R -module. For a prime ideal \mathfrak{p} of R , the residue class field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is denoted by $k(\mathfrak{p})$. For each non-negative integer i , let $H_{\mathfrak{a}}^i(M)$ denotes the i -th local cohomology module of M with respect to \mathfrak{a} ; see [1] for its definition and basic results. A Gorenstein module over a local ring R is a maximal Cohen-Macaulay module of finite injective dimension. This concept was introduced by R.Y. Sharp in [7] and studied extensively by him and other authors. In present paper, we will use the concept of relative Cohen-Macaulay modules which has been studied in [3] under the title of cohomologically complete intersections and continued in [5]. In [3] M. Hellus and P. Schenzel, as a main result, showed that if (R, \mathfrak{m}, k) is a local Gorenstein ring and \mathfrak{a} is an ideal of R with $\text{ht}_R \mathfrak{a} = c$ and $\dim R/\mathfrak{a} = d$ such that R is relative Cohen-Macaulay in $V(\mathfrak{a}) \setminus \{\mathfrak{m}\}$, then the following statements are equivalent.

- (i) $H_{\mathfrak{a}}^i(R) = 0$ for all $i \neq c$, i.e R is relative Cohen-Macaulay with respect to \mathfrak{a} .
- (ii) $H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(R)) \cong E_R(k)$ and $H_{\mathfrak{m}}^i(H_{\mathfrak{a}}^c(R)) = 0$ for all $i \neq d$.
- (iii) $\text{Ext}_R^d(k, H_{\mathfrak{a}}^c(R)) \cong k$ and $\text{Ext}_R^i(k, H_{\mathfrak{a}}^c(R)) = 0$ for all $i \neq d$.
- (iv) $\mu^i(\mathfrak{m}, H_{\mathfrak{a}}^c(R)) = \delta_{di}$.

Moreover, if \mathfrak{a} satisfies the above conditions, it follows that $\hat{R}^{\mathfrak{a}} \cong \text{Hom}_R(H_{\mathfrak{a}}^c(R), H_{\mathfrak{a}}^c(R))$ and $\text{Ext}_R^i(H_{\mathfrak{a}}^c(R), H_{\mathfrak{a}}^c(R)) = 0$ for all $i \neq 0$, where $\hat{R}^{\mathfrak{a}}$ denotes the \mathfrak{a} -adic completion of R .

As a main result, in 2.9, we generalize the above result for a Gorenstein R -module M . Indeed, it is shown that if M is a Gorenstein R -module which is relative Cohen-Macaulay in $\text{Supp}_R(M/\mathfrak{a}M) \setminus \{\mathfrak{m}\}$ such that $c = \text{ht}_M \mathfrak{a}$ and $d = \dim M/\mathfrak{a}M$, then the following statements are equivalent.

- (i) $H_{\mathfrak{a}}^i(M) = 0$ for all $i \neq c$, i.e M is relative Cohen-Macaulay with respect to \mathfrak{a} .

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- (ii) $H_m^d(H_a^c(M)) \cong E_R(k)^{r(M)}$ and $H_m^i(H_a^c(M)) = 0$ for all $i \neq d$.
- (iii) $\text{Ext}_R^d(k, H_a^c(M)) \cong k^{r(M)}$ and $\text{Ext}_R^i(k, H_a^c(M)) = 0$ for all $i \neq d$.
- (iv) $\mu^i(\mathfrak{m}, H_a^c(M)) = r(M)\delta_{di}$.

Moreover, if \mathfrak{a} satisfies the above conditions, it follows that $\text{Hom}_R(M, M) \otimes_R \hat{R}^{\mathfrak{a}} \cong \text{Hom}_R(H_a^c(M), H_a^c(M))$ and $\text{Ext}_R^i(H_a^c(M), H_a^c(M)) = 0$ for all $i \neq 0$, where $r(M)$ denotes the type of M .

2. MAIN RESULTS

Definition 2.1. We say that a finitely generated R -module M is *relative Cohen Macaulay with respect to \mathfrak{a}* if there is precisely one non-vanishing local cohomology module of M with respect to \mathfrak{a} . Clearly this is the case if and only if $\text{grade}(\mathfrak{a}, M) = \text{cd}(\mathfrak{a}, M)$, where $\text{cd}(\mathfrak{a}, M)$ is the largest integer i for which $H_a^i(M) \neq 0$. Observe that the notion of relative Cohen-Macaulay module is connected with the notion of cohomologically complete intersection ideal which has been studied in [3].

Remark 2.2. Let M be a relative Cohen-Macaulay module with respect to \mathfrak{a} and let $\text{cd}(\mathfrak{a}, M) = n$. Then, in view of [1, Theorems 6.1.4, 4.2.1, 4.3.2], it is easy to see that $\text{Supp } H_a^n(M) = \text{Supp}(M/\mathfrak{a}M)$ and $\text{ht}_M \mathfrak{a} = \text{grade}(\mathfrak{a}, M)$, where $\text{ht}_M \mathfrak{a} = \inf \{ \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Supp}(M/\mathfrak{a}M) \}$.

Definition 2.3. (See [7, Theorem 3.11].) A non-zero finitely generated module M over a local ring (R, \mathfrak{m}) is said to be a Gorenstein R -module if the following equalities hold true

$$\text{depth } M = \dim M = \text{id}_R M = \dim R.$$

Definition 2.4. For any prime ideal \mathfrak{p} of R , the number $\mu^i(\mathfrak{p}, M) := \text{vdim}_{k(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), M_{\mathfrak{p}})$ is called the i -th Bass number of M with respect to \mathfrak{p} . If (R, \mathfrak{m}) is local and M is finitely generated of depth t , then the number $\mu^t(\mathfrak{m}, M)$ is called type of M and is denoted by $r(M)$.

Let M be a non-zero finitely generated module over a local ring (R, \mathfrak{m}) and let \mathfrak{a} be an ideal of R . Let $E_R(M)$ be a minimal injective resolution for M . It is a well-known fact that $E_R(M)^i = \bigoplus_{\mathfrak{p} \in \text{Spec}(R)} \mu^i(\mathfrak{p}, M) E_R(R/\mathfrak{p})$, where $E_R(R/\mathfrak{p})$ denotes the injective hull of R/\mathfrak{p} . Now, let $c = \text{grade}(\mathfrak{a}, M)$. Then, $\Gamma_{\mathfrak{a}}(E_R(M)^i) = 0$ for all $i < c$. Therefore $H_a^c(M) = \ker(\Gamma_{\mathfrak{a}}(E_R(M))^c \rightarrow \Gamma_{\mathfrak{a}}(E_R(M))^{c+1})$. This observation provides an embedding $0 \rightarrow H_a^c(M)[-c] \rightarrow \Gamma_{\mathfrak{a}}(E_R(M))$ of complexes of R -modules where $H_a^c(M)[-c]$ is considered as a complex concentrated in homological degree zero.

Definition 2.5. (See [4, Definition 4.1].) The cokernel of the embedding $H_a^c(M)[-c] \rightarrow \Gamma_{\mathfrak{a}}(E_R(M))$ is denoted by $C_M^{\bullet}(\mathfrak{a})$ and is called the truncation complex. Therefore, there is a short exact sequence

$$0 \rightarrow H_a^c(M)[-c] \rightarrow \Gamma_{\mathfrak{a}}(E_R(M)) \rightarrow C_M^{\bullet}(\mathfrak{a}) \rightarrow 0$$

of complexes of R -modules. We observe that $H^i(C_M^{\bullet}(\mathfrak{a})) \cong H_a^i(M)$ for all $i > c$ while $H^i(C_M^{\bullet}(\mathfrak{a})) = 0$ for all $i \leq c$.

The following lemma is of assistance in the proof of the main result.

Lemma 2.6. *Assume that M is a Gorenstein module over the local ring R . Then, with the previous notations the following statements are true.*

(i) There is a short exact sequence

$$0 \rightarrow \text{Ext}_R^0(C_M(\mathfrak{a}), M) \rightarrow \text{Hom}_R(M, M) \otimes_R \hat{R}^{\mathfrak{a}} \rightarrow \text{Ext}_R^c(H_{\mathfrak{a}}^c(M), M) \rightarrow \text{Ext}_R^1(C_M(\mathfrak{a}), M) \rightarrow 0.$$

(ii) There are isomorphisms $\text{Ext}_R^{i+c}(H_{\mathfrak{a}}^c(M), M) \cong \text{Ext}_R^{i+1}(C_M(\mathfrak{a}), M)$ for all $i > 0$.

(iii) Suppose that M is relative Cohen-Macaulay with respect to \mathfrak{a} and $\text{ht}_M \mathfrak{a} = c$. Then $\text{Ext}_R^c(H_{\mathfrak{a}}^c(M), M) \cong \text{Hom}_R(M, M) \otimes_R \hat{R}^{\mathfrak{a}}$ and $\text{Ext}_R^{i+c}(H_{\mathfrak{a}}^c(M), M) = 0$ for all $i \neq 0$.

Proof. We first notice that, since M is Gorenstein, R is Cohen-Macaulay and \hat{M} is a Gorenstein \hat{R} -module. Now, let $\omega_{\hat{R}}$ be a canonical module of the ring \hat{R} . Then, in view of [2, Exercise 3.3.28], \hat{M} is isomorphic to a direct sum of finitely many copies of $\omega_{\hat{R}}$. Hence, one can use [2, Theorem 3.3.10(c)], [2, Theorem 3.3.4(d)] and the fact that $\text{Ext}_{\hat{R}}^i(\hat{M}, \hat{M}) \cong \text{Ext}_R^i(M, M) \otimes_R \hat{R}$ for all i , to see that $\text{Ext}_R^i(M, M) = 0$ for all $i > 0$ and that $\text{Hom}_R(M, M)$ is a flat R -module.

(i): Let E^\cdot be a minimal injective resolution for M . Then, since E^\cdot is a complex of injective R -modules, by applying the functor $\text{Hom}_R(-, E^\cdot)$ on the exact sequence of R -complexes in 2.5 we obtain the short exact sequence of complexes

$$0 \longrightarrow \text{Hom}_R(C_M(\mathfrak{a}), E^\cdot) \longrightarrow \text{Hom}_R(\Gamma_{\mathfrak{a}}(E^\cdot), E^\cdot) \longrightarrow \text{Hom}_R(H_{\mathfrak{a}}^c(M), E^\cdot)[c] \longrightarrow 0.$$

Now, let $\underline{x} = x_1, \dots, x_n$ be a generating set of the ideal \mathfrak{a} , and let $\check{C}_{\underline{x}}(R)$ be the Čech complex of R with respect to \underline{x} . Then, in view of [6, Theorem 1.1], there exists an isomorphism $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq \check{C}_{\underline{x}}(R) \otimes^{\mathbf{L}} M$ in the derived category. Now, since $\text{Ext}_R^i(M, M) = 0$ for all $i > 0$, we see that $\text{Hom}_R(M, M) \simeq \mathbf{R}\text{Hom}_R(M, M)$. Thus, we get the following isomorphisms

$$\begin{aligned} \mathbf{R}\text{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M), M) &\simeq \mathbf{R}\text{Hom}_R((\check{C}_{\underline{x}}(R) \otimes^{\mathbf{L}} M), M) \\ &\simeq \mathbf{R}\text{Hom}_R(\check{C}_{\underline{x}}(R), \mathbf{R}\text{Hom}_R(M, M)) \\ &\simeq \mathbf{R}\text{Hom}_R(\check{C}_{\underline{x}}(R), \text{Hom}_R(M, M)), \end{aligned}$$

in the derived category. Now, since $\text{Hom}_R(M, M)$ is flat, one can use [6, Theorem 1.1] and the above isomorphisms to deduce that $H^i(\text{Hom}_R(\Gamma_{\mathfrak{a}}(E^\cdot), E^\cdot)) \cong H^i(\mathbf{R}\text{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M), M)) = 0$ for all $i \neq 0$ and that $H^0(\text{Hom}_R(\Gamma_{\mathfrak{a}}(E^\cdot), E^\cdot)) \cong \text{Hom}_R(M, M) \otimes_R \hat{R}^{\mathfrak{a}}$. On the other hand, since $\Gamma_{\mathfrak{a}}((E^\cdot)^i) = 0$ for all $i < c = \text{grade}(\mathfrak{a}, M)$ and $\text{Hom}_R(N, X) = \text{Hom}_R(N, \Gamma_{\mathfrak{a}}(X))$ for any \mathfrak{a} -torsion R -module N and for all R -modules X , one can deduce that $\text{Ext}_R^i(H_{\mathfrak{a}}^c(M), M) = 0$ for all $i < c$. Therefore, with the aid of the above considerations, the induced long exact cohomology sequence of the above exact sequence of complexes provides the statements (i) and (ii) of the claim.

(iii): Assume that M is relative Cohen-Macaulay with respect to \mathfrak{a} and that $\text{ht}_M \mathfrak{a} = c$. Then, one can easily check that the complex $C_M(\mathfrak{a})$ is exact; and so the complex $\text{Hom}_R(C_M(\mathfrak{a}), E^\cdot)$ is also exact. Therefore, $\text{Ext}_R^i(C_M(\mathfrak{a}), M) = H^i(\text{Hom}_R(C_M(\mathfrak{a}), E^\cdot)) = 0$ for all i . Hence, the assertion follows from (i) and (ii). \square

The following lemma and definition are needed in the proof of the next theorem.

Lemma 2.7. (See [3, Proposition 4.1].) *Let n be a non-negative integer and let M be an arbitrarily R -module over a local ring (R, \mathfrak{m}) . Then the following conditions are equivalent.*

- (i) $H_{\mathfrak{m}}^i(M) = 0$ for all $i < n$.
- (ii) $\text{Ext}_R^i(R/\mathfrak{m}, M) = 0$ for all $i < n$.

Definition 2.8. Let (R, \mathfrak{m}) be a local ring and let \mathfrak{a} be a proper ideal of R . Then, we say that a non-zero R -module M is relative Cohen-Macaulay in $\text{Supp}_R(M/\mathfrak{a}M) \setminus \{\mathfrak{m}\}$, whenever $M_{\mathfrak{p}}$ is relative Cohen-Macaulay with respect to $\mathfrak{a}R_{\mathfrak{p}}$ and $\text{cd}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = c$ for all $\mathfrak{p} \in \text{Supp}_R(M/\mathfrak{a}M) \setminus \{\mathfrak{m}\}$, where $c = \text{ht}_M \mathfrak{a}$.

The following theorem is a generalization of [3, Theorem 0.1], which has been proved as a main result.

Theorem 2.9. *Let (R, \mathfrak{m}, k) be a local ring and let M be a Gorenstein R -module which is relative Cohen-Macaulay in $\text{Supp}_R(M/\mathfrak{a}M) \setminus \{\mathfrak{m}\}$ where \mathfrak{a} is an ideal of R . Set $c = \text{ht}_M \mathfrak{a}$ and $d = \dim M/\mathfrak{a}M$. Then the following statements are equivalent.*

- (i) $H_{\mathfrak{a}}^i(M) = 0$ for all $i \neq c$, i.e M is relative Cohen-Macaulay with respect to \mathfrak{a} .
- (ii) $H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(M)) \cong E_R(k)^{r(M)}$ and $H_{\mathfrak{m}}^i(H_{\mathfrak{a}}^c(M)) = 0$ for all $i \neq d$.
- (iii) $\text{Ext}_R^d(k, H_{\mathfrak{a}}^c(M)) \cong k^{r(M)}$ and $\text{Ext}_R^i(k, H_{\mathfrak{a}}^c(M)) = 0$ for all $i \neq d$.
- (iv) $\mu^i(\mathfrak{m}, H_{\mathfrak{a}}^c(M)) = r(M)\delta_{di}$.

Moreover, if one of the above statements holds, then

$$\text{Hom}_R(M, M) \otimes_R \hat{R}^{\mathfrak{a}} \cong \text{Hom}_R(H_{\mathfrak{a}}^c(M), H_{\mathfrak{a}}^c(M))$$

and $\text{Ext}_R^i(H_{\mathfrak{a}}^c(M), H_{\mathfrak{a}}^c(M)) = 0$ for all $i \neq 0$, where $\hat{R}^{\mathfrak{a}}$ denotes the \mathfrak{a} -adic completion of R .

Proof. (i) \Rightarrow (ii): First, we can use [5, Proposition 2.8] and the assumption to see that $H_{\mathfrak{m}}^i(H_{\mathfrak{a}}^c(M)) \cong H_{\mathfrak{m}}^{i+c}(M)$ for all $i \geq 0$. Now, since M is Cohen-Macaulay of dimension n , $H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(M)) \cong H_{\mathfrak{m}}^n(M)$ and $H_{\mathfrak{m}}^i(H_{\mathfrak{a}}^c(M)) = 0$ for all $i \neq d$. On the other hand, in view of [5, Theorem 2.5] and the assumption, $H_{\mathfrak{m}}^n(M)$ is an injective R -module. Therefore, one can use [5, Corollary 2.2] to see that $H_{\mathfrak{m}}^n(M) \cong E_R(k)^{r(M)}$.

(ii) \Rightarrow (iii): It follows from 2.7 that $\text{Ext}_R^i(k, H_{\mathfrak{a}}^c(M)) = 0$ for all $i < d$. On the other hand, since $H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(M))$ is injective, one can use [5, Proposition 2.1] to see that $\text{Ext}_R^i(k, H_{\mathfrak{a}}^c(M)) = 0$ for all $d < i$ and that $\text{Ext}_R^d(k, H_{\mathfrak{a}}^c(M)) \cong \text{Hom}_R(k, H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(M))) \cong k^{r(M)}$. This completes the proof. The implications (iii) \Leftrightarrow (iv) is clear.

(iii) \Rightarrow (ii): First, in view of 2.7, $H_{\mathfrak{m}}^i(H_{\mathfrak{a}}^c(M)) = 0$ for all $i < d$. Now, since $\dim_R H_{\mathfrak{a}}^c(M) \leq \dim_R M/\mathfrak{a}M = d$, one can use [1, Theorem 6.1.2] to see that $H_{\mathfrak{m}}^i(H_{\mathfrak{a}}^c(M)) = 0$ for all $d < i$. Therefore, [5, Proposition 2.1] implies that $\text{Hom}_R(R/\mathfrak{m}, H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(M))) \cong k^{r(M)}$. Hence, by [1, Theorem 7.1.2], $H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(M))$ is Artinian. Thus, one can use [5, Corollary 2.2] to see that $\mu^1(\mathfrak{m}, H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(M))) = \mu^{d+1}(\mathfrak{m}, H_{\mathfrak{a}}^c(M)) = 0$. Hence $H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(M))$ is an injective R -module. Therefore, we can use [5, Corollary 2.2] and [1, Corollary 10.2.8] to see that $H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(M)) \cong E_R(k)^{r(M)}$.

(ii) \Rightarrow (i): Since M is relative Cohen-Macaulay in $\text{Supp}_R(M/\mathfrak{a}M) \setminus \{\mathfrak{m}\}$, one can use [5, Proposition 2.8] to deduce that $H_{\mathfrak{p}R_{\mathfrak{p}}}^i(H_{\mathfrak{a}R_{\mathfrak{p}}}^c(M_{\mathfrak{p}})) \cong H_{\mathfrak{p}R_{\mathfrak{p}}}^{i+c}(M_{\mathfrak{p}})$ for all prime ideals \mathfrak{p} in

$\text{Supp}_R(M/\mathfrak{a}M) \setminus \{\mathfrak{m}\}$ and for all $i \in \mathbb{Z}$. On the other hand, since M is Gorenstein, one can use [5, Theorem 2.5] and [5, Corollary 2.2] to see that $H_{\mathfrak{m}}^n(M) \cong E_R(k)^{r(M)}$. Therefore, the assertion follows from [4, Theorem 4.4]. Next, one can use 2.6(iii) and [5, Proposition 2.1] to establish the final assertion. \square

Next, we single out a certain case of the above theorem for $M = \omega_R$, where ω_R denotes a canonical module of a Cohen-Macaulay ring R .

Corollary 2.10. *Let (R, \mathfrak{m}, k) be a local Cohen-Macaulay ring which admits a canonical R -module ω_R and let \mathfrak{a} be an ideal of R with $\text{ht}_R \mathfrak{a} = c$ and $\dim R/\mathfrak{a} = d$. Suppose that ω_R is relative Cohen-Macaulay in $V(\mathfrak{a}) \setminus \{\mathfrak{m}\}$. Then the following statements are equivalent.*

- (i) $H_{\mathfrak{a}}^i(\omega_R) = 0$ for all $i \neq c$, i.e ω_R is relative Cohen-Macaulay with respect to \mathfrak{a} .
- (ii) $H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(\omega_R)) \cong E_R(k)$ and $H_{\mathfrak{m}}^i(H_{\mathfrak{a}}^c(\omega_R)) = 0$ for all $i \neq d$.
- (iii) $\text{Ext}_R^d(k, H_{\mathfrak{a}}^c(\omega_R)) \cong k$ and $\text{Ext}_R^i(k, H_{\mathfrak{a}}^c(\omega_R)) = 0$ for all $i \neq d$.
- (iv) $\mu^i(\mathfrak{m}, H_{\mathfrak{a}}^c(\omega_R)) = \delta_{di}$.

Moreover, if \mathfrak{a} satisfies the above conditions, it follows that $\hat{R}^{\mathfrak{a}} \cong \text{Hom}_R(H_{\mathfrak{a}}^c(\omega_R), H_{\mathfrak{a}}^c(\omega_R))$ and $\text{Ext}_R^i(H_{\mathfrak{a}}^c(\omega_R), H_{\mathfrak{a}}^c(\omega_R)) = 0$ for all $i \neq 0$, where $\hat{R}^{\mathfrak{a}}$ denotes the \mathfrak{a} -adic completion of R .

Proof. We first notice that $\text{Supp}(\omega_R/\mathfrak{a}\omega_R) = V(\mathfrak{a})$. Hence, $\dim(\omega_R/\mathfrak{a}\omega_R) = \dim(R/\mathfrak{a})$ and $\text{ht}_R \mathfrak{a} = \text{ht}_{\omega_R} \mathfrak{a}$. On the other hand, one can use [2, Proposition 3.3.11] to see that ω_R is a Gorenstein R -module of type 1 and that $\text{Hom}_R(\omega_R, \omega_R) \cong R$. Hence, the assertion follows from 2.9. \square

The following corollary, which is an immediate consequence of 2.10, has been proved in [3, Theorem 0.1] as a main result.

Corollary 2.11. *Let (R, \mathfrak{m}, k) be a local Gorenstein ring and let \mathfrak{a} be an ideal of R . Set $\text{ht}_R \mathfrak{a} = c$ and $\dim R/\mathfrak{a} = d$. Suppose that R is relative Cohen-Macaulay in $V(\mathfrak{a}) \setminus \{\mathfrak{m}\}$. Then the following statements are equivalent.*

- (i) $H_{\mathfrak{a}}^i(R) = 0$ for all $i \neq c$, i.e R is relative Cohen-Macaulay with respect to \mathfrak{a} .
- (ii) $H_{\mathfrak{m}}^d(H_{\mathfrak{a}}^c(R)) \cong E_R(k)$ and $H_{\mathfrak{m}}^i(H_{\mathfrak{a}}^c(R)) = 0$ for $i \neq d$.
- (iii) $\text{Ext}_R^d(k, H_{\mathfrak{a}}^c(R)) \cong k$ and $\text{Ext}_R^i(k, H_{\mathfrak{a}}^c(R)) = 0$ for all $i \neq d$.
- (iv) $\mu^i(\mathfrak{m}, H_{\mathfrak{a}}^c(R)) = \delta_{di}$.

Moreover, if one of the above conditions is satisfied, then $\hat{R}^{\mathfrak{a}} \cong \text{Hom}_R(H_{\mathfrak{a}}^c(R), H_{\mathfrak{a}}^c(R))$ and $\text{Ext}_R^i(H_{\mathfrak{a}}^c(R), H_{\mathfrak{a}}^c(R)) = 0$ for all $i \neq 0$.

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REFERENCES

- [1] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, 1998.

- [2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1993.
- [3] M. Hellus, P. Schenzel, *On cohomologically complete intersections*, J. Algebra **320** (2008) 3733–3748.
- [4] M. Hellus, P. Schenzel, *Notes on local cohomology and duality*, arXiv:1211.4956.
- [5] M. Rahro Zargar and H. Zakeri, *On injective and Gorenstein injective dimensions of local cohomology modules*, to appear in Algebar Colloquium.
- [6] P. Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand. **92** (2003) 161–180.
- [7] R.Y. Sharp, *Gorenstein Modules*, Mathematische Zeitschrift, **115** (1970) 117–139.

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